ASYMPTOTIC THEORY OF GÖRTLER VORTICES IN THE BOUNDARY LAYER OF A LIQUID

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Hydrodynamic instability of flow in a centrifugal force field [1, 3] (Taylor–Görtler vortices) significantly affects the operation of different apparatus with curving of the flow and the initiation of the transition from the laminar into the turbulent state [3, 4]. Investigation of the development of vortices in the boundary layer of a liquid near a concave surface (Görtler vortices) has been confined for a long time, because of the complexity of the problem, to their initial linear stage. The rate of growth of the amplitude of vortices as a function of their wavelength was determined using a special form of the solution of the Navier–Stokes equations [5, 6]. The disagreement of the results of such works, especially for long-wavelength vortices, stimulated an asymptotic analysis of the development of such vortices. In [7-11] it is shown that the buildup of the boundary layer must be taken into account and the stabilizing action of nonlinear effects was studied, while in [12] the effect of the initial conditions was considered. Many results of theoretical and experimental investigations are presented in [13, 14], and (still) rare data for a gas are contained in [14, 15]. Nonstationary development of vortices is analyzed in [16], where the wavelength ranges of the most stable vortices are determined.

Asymptotic analysis of the Navier–Stokes equations for large Reynolds and Görtler numbers makes it possible to determine the basic mechanisms of development of flow instability, find the similarity parameters, and simplify the boundary-value problems, whose solution is of great theoretical and applied significance. Different aspects of the development of Görtler vortices in such an approach were studied in [17-19].

1. Consider a uniform flow of a viscous liquid with large but subcritical Reynolds number ($Re = u_\infty L/\nu = \varepsilon^{-2}$, u_∞ is the velocity of the incident flow, L is the distance along the flow from the front edge of the surface up to the point of flow instability, and ν is the coefficient of kinematic viscosity) over a concave surface. A two-dimensional laminar boundary layer forms near this surface. Under some conditions such a flow can become unstable. Then stationary Görtler vortices, extending in the longitudinal direction, arise in the boundary layer. In what follows we construct the solution of the Navier–Stokes equations for three-dimensional vortex regions in the limit $Re \to \infty$.

A schematic diagram of the flow of interest is shown in Fig. 1. Here all linear dimensions are scaled to L, R is the radius of curvature, and $\delta \sim \varepsilon$ is the characteristic thickness of the boundary layer. In what follows the velocity components u, v, and w (along the x, y, and z axes, respectively) are scaled to u_∞, the pressure p is scaled to ρu_∞^2 (ρ is the density of the liquid), and only dimensionless variables are employed. It is assumed that the curvature $k = L/R = \varkappa K < 1$, $K \sim 1$, $\varepsilon < \varkappa < 1$, i.e., the flow is considered for large Görtler numbers ($G = 2\,Re^{1/2}L/R \sim \varkappa/\varepsilon > 1$).

In Figs. 1a, b, and c are the characteristic thickness, extent, and width of the vortex regions. These quantities must be greater than the characteristic free path of molecules of the liquid $\sim\varepsilon^2$, otherwise in regions with dimensions $\Delta x \sim b$, $\Delta y \sim a$ and $\Delta z \sim c$ the Navier–Stokes equations will no longer be valid. In addition, b (vortices develop from their formation stage up to the nonlinear stage over this distance) must be greater than the transverse dimensions of these regions and must not exceed the dimensions of the body in the flow, i.e., $\varepsilon^2 < a,\ c < b \lesssim 1$.

2. In constructing the asymptotic theory of Görtler vortices, it is assumed that the instability of the boundary layer gives rise to nonlinear disturbances of the flow functions ($\Delta u \sim u$, for example) in the region where they are located, i.e., the disturbances due to the vortices affect the characteristics of the boundary layer even in the first approximation.

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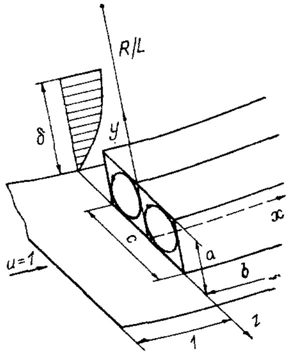

Fig. 1

In a centrifugal force field there then arises a pressure disturbance $\Delta p \sim ku^2\Delta y$, which induces a velocity $w \sim \Delta w \sim \Delta p^{1/2} \sim k^{1/2}u\Delta y^{1/2}$. Since estimates were obtained for Δp and w by comparing the orders of magnitudes of the convective terms of the Navier-Stokes equations, the mechanism of convection is the main mechanism for formation of the vortices.

Let the vortices appear near the wall in the part of the boundary layer with characteristic thickness $\Delta y \sim a < \delta \sim \varepsilon$, where the vorticity of the flow is highest and the velocity u is proportional to the distance from the surface $u \sim \Delta y/\varepsilon$. Then the following estimates are valid:

$$u \sim a/\varepsilon, \quad \Delta p \sim \varkappa a^3/\varepsilon^2, \quad w \sim \varkappa^{1/2}a^{3/2}/\varepsilon. \tag{2.1}$$

In the general case the thickness and width of the vortex region are of the same order of magnitude $\Delta y \sim a \sim \Delta z \sim c$, and for this reason

$$a \sim c \sim \varkappa b^2. \tag{2.2}$$

The estimates (2.1) and (2.2) make it possible to introduce for the region 3 near the wall variables and asymptotic expansions of the flow functions (here the disturbed regions of the flow are enumerated in the manner traditionally employed in investigation performed in this country (see, for example, [20]):

$$\begin{aligned} &x = bx_3, \quad y = \varkappa b^2 y_3, \quad z = \varkappa b^2 z_3, \\ &u = (\varkappa b^2/\varepsilon)u_3 + \ldots, \quad v = (\varkappa^2 b^3/\varepsilon)v_3 + \ldots, \\ &w = (\varkappa^2 b^3/\varepsilon)w_3 + \ldots, \quad \Delta p \approx (\varkappa^4 b^6/\varepsilon^2)p_3 + \ldots \end{aligned} \tag{2.3}$$

(Δp is determined relative to the pressure on the surface at the point of formation of the vortices). Substituting Eqs. (2.3) into the Navier–Stokes equations and passing to the limit $\varepsilon \to 0$, $\varepsilon < \varkappa < 1$ show that in the general case for $b \sim (\varepsilon/\varkappa)^{3/5} < 1$ the development of vortices is approximately described by Navier–Stokes equations, parabolized in the longitudinal direction, without a longitudinal pressure gradient:

$$\begin{aligned} &\partial u_3/\partial x_3 + \partial v_3/\partial y_3 + \partial w_3/\partial z_3 = 0, \\ &u_3\partial u_3/\partial x_3 + v_3\partial u_3/\partial y_3 + w_3\partial u_3/\partial z_3 = \partial^2 u_3/\partial y_3^2 + \partial^2 u_3/\partial z_3^2, \\ &u_3\partial v_3/\partial x_3 + v_3\partial v_3/\partial y_3 + w_3\partial v_3/\partial z_3 + Ku_3^2 + \partial p_3/\partial y_3 = \partial^2 v_3/\partial y_3^2 + \partial^2 v_3/\partial z_3^2, \\ &u_3\partial w_3/\partial x_3 + v_3\partial w_3/\partial y_3 + w_3\partial w_3/\partial z_3 + \partial p_3/\partial z_3 = \partial^2 w_3/\partial y_3^2 + \partial^2 w_3/\partial z_3^2. \end{aligned} \tag{2.4}$$

The conditions of attachment and impermeability are satisfied on the surface:

$$u_3 = v_3 = w_3 = 0 \quad (y_3 = 0), \tag{2.5}$$

and the external and initial conditions are obtained by joining with the flow in the part of the boundary layer near the wall:

$$\begin{aligned} &u_3 \to Ay_3, \quad v_3, \ w_3 \to 0, \\ &p_3 \to -KA^2y_3^3/3 \quad (x_3 \to -\infty \ \text{ or } \ y_3 \to \infty). \end{aligned} \tag{2.6}$$

Here $A = (\partial u_0/\partial y_2)_w$; $u_0(y_2)$ is the profile of the velocity u in the boundary layer at the point of formation of the vortices; $y_2 = y/\varepsilon$. The condition of periodicity is satisfied in the transverse direction:

$$u_3,\ v_3,\ w_3,\ p_3(x_3,\ y_3,\ z_3) = u_3,\ v_3,\ w_3,\ p_3(x_3,\ y_3,\ z_3 + \lambda) \tag{2.7}$$

where λ is the wavelength of the vortices.

The boundary-value problem (2.4)-(2.7) describes the nonlinear development of short-wavelength Görtler vortices with $a \sim c \sim \varepsilon^{6/5}/\varkappa^{1/5} < \delta \sim \varepsilon$ in the part of the boundary layer near the wall. The quantity A is the only characteristic of the boundary layer that appears here and the vortices develop in a plane-parallel flow, since the flow functions in the boundary layer do not change significantly over a longitudinal distance $\Delta x \sim b \sim (\varepsilon/\varkappa)^{3/5} < 1$.

For $(\varepsilon/\varkappa)^{3/5} < b < (\varepsilon/\varkappa)^{1/2}$ the dissipative terms in Eq. (2.4) are not significant and the condition of impermeability is satisfied on the surface. In order to satisfy the conditions of attachment, it is necessary to consider a viscous boudnary layer. If, however, $\varepsilon/\varkappa^{1/2} < b < (\varepsilon/\varkappa)^{3/5}$, then the convective terms in Eq. (2.4) are not significant. But then the mechanism of flow instability disappears and for such vortices the estimates (2.1) and (2.2) are not valid.

The variables x_3, y_3, z_3, u_3, v_3, w_3 and p_3 can be scaled to $(\lambda/2\pi K)^{1/2}$, $\lambda/2\pi$, $\lambda/2\pi$, $A\lambda/2\pi$, $AK^{1/2}(\lambda/2\pi)^{3/2}$, $AK^{1/2}(\lambda/2\pi)^{3/2}$ and $A^2K\lambda/2\pi$, respectively. Then Eqs. (2.4)-(2.7) assume the form

$$\begin{gathered}\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0,\\ \mathrm{Re}_1(u\partial u/\partial x + v\partial u/\partial y + w\partial u/\partial z) = \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2,\\ \mathrm{Re}_1(u\partial v/\partial x + v\partial v/\partial y + w\partial v/\partial z + u^2 + \partial p/\partial y) = \partial^2 v/\partial y^2 + \partial^2 v/\partial z^2,\\ \mathrm{Re}_1(u\partial w/\partial x + v\partial w/\partial y + w\partial w/\partial z + \partial p/\partial z) = \partial^2 w/\partial y^2 + \partial^2 w/\partial z^2,\\ u = v = w = 0\ (y = 0),\ u \to y,\ v,\ w \to 0,\ p \to -y^3/3\ (x \to -\infty\\ \text{or}\quad y \to \infty),\ u,\ v,\ w,\ p(x,\ y,\ z) = u,\ v,\ w,\ p(x,\ y,\ z + 2\pi),\\ \mathrm{Re}_1 = AK^{1/2}(\lambda/2\pi)^{5/2} \sim 1,\end{gathered} \tag{2.8}$$

where, for brevity, the index 3 in the variables is dropped and Re_1 is the local Reynolds number.

For small disturbances of the boundary layer Eqs. (2.8) can be linearized relative to the initial conditions:

$$u = y + \alpha U + \ldots,\ v = \alpha V + \ldots,\ w = \alpha W + \ldots,$$
$$p = -y^3/3 + \alpha P + \ldots,\ \alpha < 1.$$

Representing the linearized solution (see, for example, [5]) as

$$\begin{gathered}U = \exp(\beta x) U_1(y) \cos z,\quad V = \exp(\beta x) V_1(y) \cos z,\\ W = \exp(\beta x) W_1(y) \sin z,\quad P = \exp(\beta x) P_1(y) \cos z\end{gathered} \tag{2.9}$$

the boudnary-value problem (2.8) can be reduced to the following problem:

$$\begin{gathered}\beta U_1 + dV_1/dy + W_1 = 0,\ \mathrm{Re}_1(\beta y U_1 + V_1) = d^2U_1/dy^2 - U_1,\\ \mathrm{Re}_1(\beta y V_1 + 2yU_1 + dP_1/dy) = d^2V_1/dy^2 - V_1,\\ \mathrm{Re}_1(\beta y W_1 - P_1) = d^2W_1/dy^2 - W_1,\\ U_1(0) = V_1(0) = W_1(0) = U_1(\infty) = V_1(\infty) = W_1(\infty) = 0.\end{gathered} \tag{2.10}$$

This is a typical eigenvalue problem: for each value of the growth rate β of the amplitude of the vortices it is necessary to find one or several values of Re_1 for which a nontrivial solution of the problem exists. Equation (2.10) was solved with the help of the method of backward iterations [21]. Figure 2 shows β_1 versus Re_1 for the first vortex mode of the curve 1, which passes practically through the origin; it is linear up to $\mathrm{Re}_1 \approx 0.3$ and As Re_1 increases further the growth rate β_1 increases monotonically and approaches its limit for $\mathrm{Re}_1 \to \infty$:

$$\beta_n = n^{-1/2}, \; n = 1, 2, 3, \ldots, \tag{2.11}$$

where n is the number of the mode. The relation (2.11) can be obtained from Eq. (2.10) is an analytical form, if the right-hand sides are dropped there [22]. An analogous limit can be obtained with the help of Eq. (2.3) with $(\varepsilon/\varkappa)^{3/5} < b < (\varepsilon/\varkappa)^{1/2}$.

Figure 3 shows the profiles $U_1(y)$ for $Re_1 = 0.864$, 2.64, and 114 (curves 1-3). For large values of Re_1 the maximum of $U_1(y)$ lies close to the surface, and in the limit a viscous sublayer is formed here. As Re_1 decreases the maximum of $U_1(y)$ moves away from the surface, i.e., the vortices "float up."

The growth in the amplitude of the vortices is characterized by the product

$$\beta x = BX/L, \; X \sim L,$$
$$B = (\varkappa/\varepsilon)^{3/5}\beta A^{1/5}K^{3/5}/\mathrm{Re}_1^{1/5} = (\varkappa/\varepsilon)^{3/5}\beta(2\pi K/\lambda)^{1/2}. \tag{2.12}$$

Curve 2 in Fig. 2 shows the change in the quantity $\beta_1/\mathrm{Re}_1^{1/5}$, which has a maximum at $Re_1 \approx 4.4$.

3. The solution of the boundary-value problem (2.10) does not permit determining the final value of Re_1 for neutral vortices, i.e, for $\beta = 0$. As λ decreases, R_1 decreases correspondingly. But then the convective terms become insignificant in the equations of motion and the mechanism of instability disappears. For this reason, a decrease in λ must be compensated by an increase in the characteristic velocity, so that the value of the local Reynolds number would remain finite. This is possible if the vortices "float up." Since the friction stress remains of the same order of magnitude in the entire boundary layer and the flow in the disturbed region "floating up" to a characteristic height h is viscous, we obtain

$$a \sim \varepsilon b^{1/3}, \; c \sim (\varepsilon^3/\varkappa b)^{1/2}, \; h \sim \varkappa b^2, \; (\varepsilon/\varkappa)^{3/5} < b \leqslant (\varepsilon/\varkappa)^{1/2},$$
$$u \sim h/\varepsilon, \; \Delta u \sim a/\varepsilon, \; v \sim ah/\varepsilon b, \; w \sim ch/\varepsilon b, \; \Delta p \sim w^2 \sim (ch/\varepsilon b)^2,$$

which makes it possible to introduce the following variables and asymptotic expansions:

$$x = bx_3, \; y = \varkappa b^2 y_2 + \varepsilon b^{1/3} y_3, \; z = (\varepsilon^3/\varkappa b)^{1/2} z_3, \tag{3.1}$$
$$u = (\varkappa b^2/\varepsilon) u_0(y_2) + b^{1/3} u_3 + \ldots, \; v = \varkappa b^{4/3} v_3 + \ldots,$$
$$w = (\varkappa\varepsilon b)^{1/2} w_3 + \ldots, \; \Delta p \approx -(\varkappa^4 b^6/\varepsilon^2) K \int_0^{y_2} u_0^2 dy_2 + \varkappa\varepsilon b p_3 + \ldots.$$

Here $y_2 \sim h/\varkappa b^2 \sim 1$ is the height up to which the vortex "floats," and for $h \sim \varepsilon$, y_2 is the vertical variable of the boundary layer. Substituting Eqs. (3.1) into the Navier–Stokes equations and passing to the limit $\varepsilon \to 0$, $\varepsilon < \varkappa < 1$, $(\varepsilon/\varkappa)^{3/5} < b \leqslant (\varepsilon/\varkappa)^{1/2}$ show that to a first approximation the following equations are valid for the disturbed region:

$$\partial v_3/\partial y_3 + \partial w_3/\partial z_3 = 0, \tag{3.2}$$
$$u_0 \partial u_3/\partial x_3 + v_3(du_0/dy_2 + \partial u_3/\partial y_3) + w_3 \partial u_3/\partial z_3 = \partial^2 u_3/\partial z_3^2,$$
$$u_0 \partial v_3/\partial x_3 + v_3 \partial v_3/\partial y_3 + w_3 \partial v_3/\partial z_3 + 2K u_0 u_3 = \partial^2 v_3/\partial z_3^2,$$
$$u_0 \partial w_3/\partial x_3 + v_3 \partial w_3/\partial y_3 + w_3 \partial w_3/\partial z_3 + \partial p_3/\partial z_3 = \partial^2 w_3/\partial z_3^2.$$

The solution of this system must decay upstream and from top to bottom as a function of the coordinate y_2:

$$u_3, \; v_3, \; w_3, \; p_3 \to 0 \quad (x_3 \to -\infty \;\text{ or }\; y_3 \to \pm\infty), \tag{3.3}$$

and it must also satisfy the periodicity condition (2.7).

The boundary-value problem (2.7), (3.2), and (3.3) describes the nonlinear development of Görtler vortices which have floated up. Its solution determines the small changes in the velocity u and the finite gradients of the velocity $\partial u/\partial y$. The only characteristics of the boundary layer that appear here are the values of u_0 and du_0/dy_2 at the height y_2 to which the vortices rise, and the vortices develop over the short distance $(\varepsilon/\varkappa)^{3/5} < \Delta x \sim b \lesssim (\varepsilon/\varkappa)^{1/2} < 1$ in a flow with constant velocity.

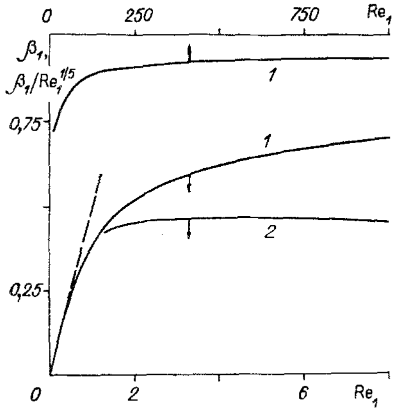

Fig. 2

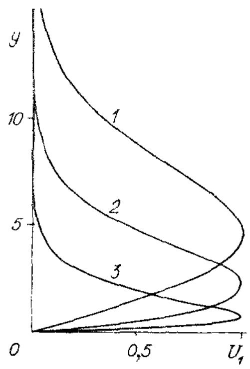

Fig. 3

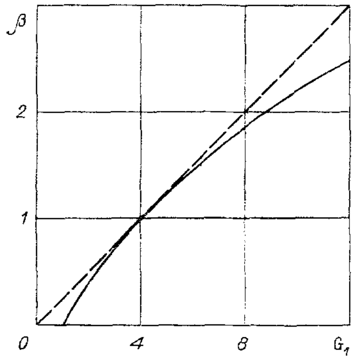

Fig. 4

The variables x_3, y_3, z_3, u_3, v_3, w_3 and p_3 can be scaled to $u_0(\lambda/2\pi)^2$, $(u_0/(du_0/dy_2))^{1/3}(\lambda/2\pi)^{2/3}$, $\lambda/2\pi$, $u_0^{1/3}(du_0/dy_2)^{2/3}(\lambda/2\pi)^{2/3}$, $(u_0/(du_0/dy_2))^{1/3}(2\pi/\lambda)^{4/3}$, $2\pi/\lambda$ and $(2\pi/\lambda)^2$, respectively. Then Eqs. (2.7), (3.2), and (3.3) assume the form

$$
\begin{aligned}
&\partial v/\partial y + \partial w/\partial z = 0,\\
&\partial u/\partial x + v(1 + \partial u/\partial y) + w\partial u/\partial z = \partial^2 u/\partial z^2,\\
&\partial v/\partial x + v\partial v/\partial y + w\partial v/\partial z + G_1 u = \partial^2 v/\partial z^2,\\
&\partial w/\partial x + v\partial w/\partial y + w\partial w/\partial z + \partial p/\partial z = \partial^2 w/\partial z^2,\\
&u,\ v,\ w,\ p \to 0\ \ (x \to -\infty \ \text{ or } \ y \to \pm\infty),\\
&u,\ v,\ w,\ p(x,\ y,\ z) = u,\ v,\ w,\ p(x,\ y,\ z + 2\pi),\\
&G_1 = 2Ku_0(du_0/dy_2)(\lambda/2\pi)^4 \sim 1,
\end{aligned}
\tag{3.4}
$$

where for brevity the index in the variables has been dropped and G_1 is the local Görtler number. Linearization is possible for small amplitudes of the vortices, when the quadratic terms in Eqs. (3.4) are smaller than the linear terms. The representation (2.9) leads to the conclusion that a nontrivial solution (solid curve in Fig. 4, the dashed curve shows the tangent to it, passing through the origin $\beta = G_1/4$) exists only if

$$G_1 = (1 + \beta_n)^2,\ n = 1,\ 2,\ 3,\ldots\ . \tag{3.5}$$

The increase in the amplitude of the vortices in the case under consideration is characterized by $B = (\beta/bu_0)(2\pi/\lambda)^2$. As $b \to (\varepsilon/\varkappa)^{3/5}$, i.e., passing to vortices in the part of the boundary layer near the wall, the expressions for G_1 and B can be represented as

$$
\begin{aligned}
&G_1 \to 2\,\mathrm{Re}_1^2 y_2/(\lambda/2\pi), \quad y_2 \to \lambda/2\pi,\\
&B \to (\varkappa/\varepsilon)^{3/5}(\beta/\mathrm{Re}_1)\left(A^{1/5}K^{3/5}/\mathrm{Re}_1^{1/5}\right)(\lambda/2\pi)/y_2.
\end{aligned}
$$

Comparing Eqs. (2.12) and (3.6) shows that the dashed curve (Fig. 4) in the variables of Fig. 2 is determined by the relation $\beta = \mathrm{Re}_1/2$ and is the asymptote to the curve 1 in the limit

$Re_1 \to 0$, i.e., as Re_1 decreases and the vortices "float up" the curves of all β_n ($n \geq 1$) versus Re_1 must approach the linear section of curve 1 and merge with it. Its continuation in the limit $Re_1 \to 0$ is the dashed line, which transforms, as the scales β and Re_1 change, into the dashed line in Fig. 4 and then joins smoothly with the solid line at finite values of G_1.

The relation (3.5) determines the quantity $G_1 = 1$, when all modes become neutral ($\beta_n = 0$, $n \geq 1$). The product $u_0(du_0/dy_2)$, evidently, has a maximum, and the corresponding value of y_2 characterizes the height to which the neutral vortices with the shortest wavelength "rise."

4. The estimates (2.1) and (2.2) remain valid as the transverse dimensions of the vortex region increase, right up to $a \sim c \sim \delta \sim \varepsilon$. Then the vortices will fill most of the boundary layer (region 2) with characteristic thickness $\Delta y \sim \delta \sim \varepsilon$, for which the following variables and asymptotic expansions are introduced:

$$x = (\varepsilon/\varkappa)^{1/2}x_2,\ \ y = \varepsilon y_2,\ \ z = \varepsilon z_2,\ \ \Delta p \approx \varkappa\varepsilon p_2 + \ldots,$$
$$u = u_2 + \ldots,\ \ v = \varkappa^{1/2}\varepsilon^{1/2}v_2 + \ldots,\ \ w = \varkappa^{1/2}\varepsilon^{1/2}w_2 + \ldots .$$

Substituting these expansions into the Navier–Stokes equations and passing to the limit $\varepsilon \to 0$, $\varepsilon < \kappa < 1$, we find that to a first approximation the flow is described by the Euler equations without the longitudinal pressure gradient:

$$\begin{aligned} &\partial u_2/\partial x_2 + \partial v_2/\partial y_2 + \partial w_2/\partial z_2 = 0,\\ &u_2\partial u_2/\partial x_2 + v_2\partial u_2/\partial y_2 + w_2\partial u_2/\partial z_2 = 0,\\ &u_2\partial v_2/\partial x_2 + v_2\partial v_2/\partial y_2 + w_2\partial v_2/\partial z_2 + Ku_2^2 + \partial p_2/\partial y_2 = 0,\\ &u_2\partial w_2/\partial x_2 + v_2\partial w_2/\partial y_2 + w_2\partial w_2/\partial z_2 + \partial p_2/\partial z_2 = 0. \end{aligned} \tag{4.1}$$

Only the impermeability condition can be satisfied at the surface

$$v_2 = 0 \quad (y_2 = 0), \tag{4.2}$$

and the initial and external boundary conditions are obtained by joining with the solution for the entire boundary layer:

$$u_2 \to u_0(y_2),\ \ p_2 \to -K\int_0^{y_2} u_0^2 dy_2,\ \ v_2,\ w_2 \to 0\ \ (x_2 \to -\infty \ \text{ or } \ y_2 \to \infty). \tag{4.3}$$

The solution of Eqs. (4.1) must also satisfy the periodicity conditions (2.7).

Among the characteristics of the boundary layer, the velocity profile $u_0(y_2)$ enters into the boundary-value problem (2.7) and (4.1)-(4.3). Here $\Delta x \sim (\varepsilon/\varkappa)^{1/2} < 1$ and the longitudinal change in the flow functions in the boundary layer is insignificant. A viscous boundary layer can be considered near the surface in order to satisfy the attachment conditions.

For the case when λ does not exceed the thickness δ_1 of the boundary layer (the value of the coordinate y_2 for which $u_0 = 0.99$ can be taken for δ_1), the variables x_2, y_2, z_2, v_2, w_2 and p_2 are scaled to $(\lambda/2\pi K)^{1/2}$, $\lambda/2\pi$, $\lambda/2\pi$, $(\lambda K/2\pi)^{1/2}$, $(\lambda K/2\pi)^{1/2}$ and $\lambda K/2\pi$, respectively. In the new variables (without the index 2) Eqs. (2.7) and (4.1)-(4.3) assume the form

$$\begin{aligned} &\partial u/\partial x + \partial v/\partial y + \gamma_3\partial w/\partial z = 0,\\ &u\partial u/\partial x + v\partial u/\partial y + \gamma_3 w\partial u/\partial z = 0,\\ &\gamma_2\gamma_3(u\partial v/\partial x + v\partial v/\partial y + \gamma_3 w\partial v/\partial z) + u^2 + \partial p/\partial y = 0,\\ &u\partial w/\partial x + v\partial w/\partial y + \gamma_3 w\partial w/\partial z + \partial p/\partial z = 0,\ \ v = 0\ \ (y = 0),\\ &u \to u_0(\gamma_1 y),\ \ p \to -\int_0^y u_0^2 dy,\ v,\ w \to 0\ \ (x \to -\infty \ \text{ or } \ y \to \infty),\\ &u,\ v,\ w,\ p(x,\ y,\ z) = u,\ v,\ w,\ p(x,\ y,\ z + 2\pi), \end{aligned} \tag{4.4}$$

where $\gamma_1 = \lambda/2\pi\delta_1$; $0 < \gamma_1 \leq 1$; $\gamma_2 = \gamma_3 = 1$. For small disturbances of the boundary layer Eqs. (4.4) can be linearized with respect to the initial conditions. The representation (2.9) makes it possible to obtain for a boundary layer with intense suction [23]

$$u_0 = 1 - \exp(-\gamma_1 y),\ v_0 = -1/\delta_1,\ \delta_1 = \text{const}, \tag{4.5}$$

an analytical solution and the following expression for β [22]:

$$\beta_n^{-2} = \gamma_1 (n^2 - 1)/2 + n,\ n = 1, 2, 3, \ldots. \tag{4.6}$$

As $\gamma_1 \to 0$ the expression (4.6) transforms into the expression (2.11). The growth in the amplitude of the vortices is characterized, in this case, by the quantity

$$B = (\beta/b)(2\pi K/\lambda)^{1/2},\ (\varepsilon/\varkappa)^{3/5} < b \leqslant (\varepsilon/\varkappa)^{1/2}. \tag{4.7}$$

5. In the study of long-wavelength vortices ($c > \delta \sim \varepsilon$) it is assumed that the nonlinear changes occur in the main part of the boundary layer (region 2) with characteristic thickness $\Delta y_2 \sim \delta \sim \varepsilon$. Above this region there can be a weakly disturbed zone of the external flow (region 1) with characteristic thickness $\Delta y_1 \sim c$. The following estimates can be obtained for the region 2:

$$\begin{gathered}\Delta x_2 \sim b,\ \Delta y_2 \sim a \sim \varepsilon,\ \Delta z_2 \sim c \sim \varkappa^{1/2}\varepsilon^{1/2}b,\ (\varepsilon/\varkappa)^{1/2} < b \leqslant 1,\\ u_2 \sim \Delta u_2 \sim 1,\ v_2 \sim \varepsilon/b,\ w_2 \sim \varkappa^{1/2}\varepsilon^{1/2},\ \Delta p_2 \sim \varkappa\varepsilon.\end{gathered} \tag{5.1}$$

In order for an interaction to exist between regions 1 and 2 it is necessary that the velocity v in them remain of the same order of magnitude. Then, for region 1 the following estimates are valid:

$$\begin{gathered}\Delta x_1 \sim b,\ \Delta y_1 \sim \Delta z_1 \sim \varkappa^{1/2}\varepsilon^{1/2}b,\ u_1 \sim 1,\ v_1 \sim w_1 \sim \varepsilon/b,\\ \Delta u_1 \sim \Delta p_1 \sim \varkappa^{1/2}\varepsilon^{3/2}/b.\end{gathered} \tag{5.2}$$

Comparing the estimates (5.1) and (5.2) shows that $\Delta p_1/\Delta p_2 \sim (\varepsilon/\varkappa)^{1/2}/b < 1$, i.e., the disturbances induced in region 2 decay in the region 1 and there is no back effect on it.

The following variables and asymptotic expansions are introduced for region 2 on the basis of the estimates (5.1):

$$\begin{gathered}x = bx_2,\ y = \varepsilon y_2,\ z = \varkappa^{1/2}\varepsilon^{1/2}bz_2,\ \Delta p \approx \varkappa\varepsilon p_2 + \ldots,\\ u = u_2 + \ldots,\ v = (\varepsilon/b)v_2 + \ldots,\ w = \varkappa^{1/2}\varepsilon^{1/2}w_2 + \ldots,\end{gathered}$$

substituting which into the Navier–Stokes equations and passing to the limit $\varepsilon \to 0$, $\varepsilon < \varkappa < 1$, $(\varepsilon/\varkappa)^{1/2} < b \lesssim 1$ we find that, to a first approximation, the flow is described by the equations

$$\begin{gathered}\partial u_2/\partial x_2 + \partial v_2/\partial y_2 + \partial w_2/\partial z_2 = 0,\ \ Ku_2^2 + \partial p_2/\partial y_2 = 0,\\ u_2\partial u_2/\partial x_2 + v_2\partial u_2/\partial y_2 + w_2\partial u_2/\partial z_2 = b\partial^2 u_2/\partial y_2^2,\\ u_2\partial w_2/\partial x_2 + v_2\partial w_2/\partial y_2 + w_2\partial w_2/\partial z_2 + \partial p_2/\partial z_2 = b\partial^2 w_2/\partial y_2^2.\end{gathered} \tag{5.3}$$

The conditions

$$u_2 = w_2 = 0,\ v_2 = bv_{0w}\quad (y_2 = 0), \tag{5.4}$$

must be satisfied on the surface and the initial and external boundary conditions are found by joining with the solutions for the boundary layer and the external flow:

$$\begin{gathered}u_2 = u_0(y_2),\ v_2 = bv_0(y_2),\ w_2 = 0,\\ p_2 = -K\int_0^{y_2} u_0^2 dy_2\quad (x_2 = -x_0/b),\\ u_2 \to 1,\ w_2 \to 0,\ p_2 \to -K\int_0^{y_2} u_2^2 dy_2\quad (y_2 \to \infty).\end{gathered} \tag{5.5}$$

In addition, the periodicity condition (2.7) must be satisfied.

The boundary-value problem (2.7) and (5.3)-(5.5) describes the nonlinear development of long-wavelength vortices. For b < 1 Eq. (5.3) does not contain any dissipative terms. The initial profile $u_0(y_2)$ is the only characteristic of the boundary layer that appears in the problem; the longitudinal change of the profile is not significant. In order to satisfy the conditions of attachment a viscous boundary layer can be studied near the surface. If, however, $b \sim 1$, the longitudinal scales of the boundary layer and of the vortices are the same and the longitudinal change in the flow functions in the boundary layer must be taken into account.

It is helpful also to scale the variables x_2, y_2, z_2, v_2, w_2 and p_2 to $\lambda/2\pi K^{1/2}\delta_1^{1/2}$, δ_1, $\lambda/2\pi$, $2\pi K^{1/2}\delta_1^{3/2}/\lambda$, $K^{1/2}\delta_1^{1/2}$ and $K\delta_1$. In the new variables Eqs. (2.7) and Eqs. (5.3)-(5.5) for $b \sim 1$ assume the form

$$\begin{gathered}\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0,\ u^2 + \partial p/\partial y = 0,\\ \mathrm{Re}_2(u\partial u/\partial x + v\partial u/\partial y + w\partial u/\partial z) = \partial^2 u/\partial y^2,\\ \mathrm{Re}_2(u\partial w/\partial x + v\partial w/\partial y + w\partial w/\partial z + \partial p/\partial z) = \partial^2 w/\partial y^2,\\ u = w = 0,\ v = (\delta_1/\mathrm{Re}_2)v_{0w} \quad (y = 0),\\ u \to 1,\ w \to 0,\ p \to -\int_0^y u^2 dy \quad (y \to \infty),\\ u = u_0,\ v = (\delta_1/\mathrm{Re}_2)v_0,\ w = 0,\\ p = -\int_0^y u_0^2 dy \quad \left(x = -\left(\mathrm{Re}_2/\delta_1^2\right)x_0\right),\\ u,\ v,\ w,\ p(x,\ y,\ z) = u,\ v,\ w,\ p(x,\ y,\ z + 2\pi),\\ \mathrm{Re}_2 = 2\pi K^{1/2}\delta_1^{5/2}/\lambda \sim 1,\end{gathered} \tag{5.6}$$

where for brevity the index 2 in the variables is dropped; Re_2 is the local Reynolds number. The corresponding boundary-value problem for $b < 1$ can be obtained from Eqs. (5.6) by passing to the limit $\mathrm{Re}_2 \to \infty$.

For the case when $a \sim c \sim \delta \sim \varepsilon$ but λ is greater than δ_1, the use of the variables (5.6) makes it possible to represent the boundary-value problem in the form (4.4) with $\gamma_1 = \gamma_3 = 1$, $\gamma_2 = (2\pi\delta_1/\lambda)^2$, $0 < \gamma_2 \leq 1$. Its solution for $\gamma_2 \to 0$ must be the limiting solution for (5.6) in the limit $\mathrm{Re}_2 \to \infty$. In the linear approximation, using the representation (2.9) and the profile (4.5), a solution of the problem and the relation for β can be found analytically:

$$\beta_n^{-2} = (n^2 - 1)/2 + n\gamma_2^{1/2},\quad n = 1, 2, 3, \ldots \tag{5.7}$$

It is obvious that for $\gamma_1 = \gamma_2 = 1$ the expressions (4.6) and (5.7) are identical. It is remarkable that for the first mode $\beta_1 = \gamma_2^{-1/4}$ and $\beta_1 \to \infty$ as $\gamma_2 \to 0$, while for all other modes β_n remains finite as $\gamma_2 \to 0$, i.e., the first mode builds up linearly over shorter distances than all subsequent modes.

For small disturbances of the boundary layer the problem (5.6) can be linearized relative to the initial conditions. The representation (2.9) makes it possible to derive a system of ordinary differential equations and boundary conditions:

$$\begin{gathered}\beta U_1 + dV_1/dy + W_1 = 0,\ 2u_0 U_1 + dP_1/dy = 0,\\ \mathrm{Re}_2\ (\beta u_0 U_1 + U_1\partial u_0/\partial x + V_1\partial u_0/\partial y) + \delta_1 v_0 dU_1/dy = d^2U_1/dy^2,\\ \mathrm{Re}_2(\beta u_0 W_1 - P_1) + \delta_1 v_0 dW_1/dy = d^2 W_1/dy^2,\\ U_1(0) = V_1(0) = W_1(0) = U_1(\infty) = W_1(\infty) = P_1(\infty) = 0.\end{gathered} \tag{5.8}$$

A numerical solution of the problem (5.8) was found for the profile (4.5) with $\gamma_1 = 1$ [21]. Figure 5 shows β_2 versus Re_2 (curve 2). One can see that $\beta_2 = 0$ for $\mathrm{Re}_2 \approx 2.32$, i.e., the second mode becomes neutral. As Re_2 increases, β_2 approaches its asymptotic value $(2/3)^{1/2} \approx 0.816$ (see the expression (5.7) for $n = 2$ and $\gamma_2 \to 0$). For the vortex development regime under study, the growth in the amplitude of the vortices is characterized by the quantity

$$\begin{gathered}B = (\beta/b)(2\pi K/\lambda)^{1/2}\gamma_2^{1/4} = \beta\,\mathrm{Re}_2/b\delta_1^2,\\ (\varepsilon/\varkappa)^{1/2} < b \leqslant 1,\ 0 < \gamma_2 < 1.\end{gathered} \tag{5.9}$$

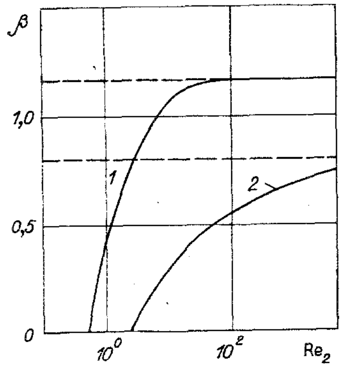

Fig. 5

As $b \to (\varepsilon/\varkappa)^{1/2}$ and $\gamma_2 \to 1$ the expressions (4.7) and (5.9) become identical.

6. We now consider the linear development of long-wavelength vortices, which are now no longer localized inside the boundary layer. In this case the region 1 is perturbed and the region 2 is weakly perturbed. Near the surface there must be a region 3 which, in the general case, is viscous and nonlinear and for which the following estimates are valid:

$$\begin{gathered}\Delta x_3 \sim b,\ \Delta y_3 \sim \varepsilon b^{1/3},\ \Delta z_3 \sim c,\ v_3 \sim \varepsilon/b^{1/3},\\ u_3 \sim \Delta u_3 \sim b^{1/3},\ w_3 \sim c/b^{2/3},\ \Delta p_3 \sim c^2/b^{4/3}.\end{gathered} \tag{6.1}$$

In the region 2 Δp_2 is produced by centrifugal effects and $\Delta p_2 \sim \Delta p_3$. For this reason here the following estimates are valid:

$$\begin{gathered}\Delta x_2 \sim b,\ \Delta y_2 \sim \varepsilon,\ \Delta z_2 \sim c,\ u_2 \sim 1,\ \Delta u_2 \sim c^2/\varkappa\varepsilon b^{4/3},\\ v_2 \sim c^2/b^{7/3},\ w_2 \sim c/b^{1/3},\ \Delta p_2 \sim c^2/b^{4/3}.\end{gathered} \tag{6.2}$$

In the region 1 all perturbations should decay and therefore

$$\begin{gathered}\Delta x_1 \sim b,\ \Delta y_1 \sim \Delta z_1 \sim c,\ u_1 \sim 1,\ \Delta u_1 \sim \Delta p_1 \sim c^2/b^{4/3},\\ v_1 \sim w_1 \sim c/b^{1/3}.\end{gathered} \tag{6.3}$$

When regions 1 and 2 interact $v_1 \sim v_2$. One can see from Eqs. (6.2) and (6.3) that this is possible if

$$c \sim \varkappa b^2,\ b > \varepsilon^{1/2}/\varkappa^{1/2}. \tag{6.4}$$

On the other hand, when regions 2 and 3 interact nontrivial joining of the expansions for the velocity $u(\Delta u_2 \sim u_3)$ is necessary; this is possible if (see Eqs. (6.1) and (6.2))

$$c \sim \varkappa^{1/2}\varepsilon^{1/2}b^{5/6},\ b \leqslant 1. \tag{6.5}$$

In the general case, when all three regions interact with one another and a completely three-layer structure of the disturbed flow is realized near the concave surface [24, 25], the estimates (6.4) and (6.5) and

$$b \sim \varepsilon^{3/7}/\varkappa^{3/7},\ c \sim \varkappa^{1/7}\varepsilon^{6/7} \tag{6.6}$$

are valid.

We give below the final form of the boundary-value problems for the regions 1 and 3 for the general case (6.6):

$$\begin{gathered}\partial^2 p_1/\partial y_1^2 + \partial^2 p_1/\partial z_3^2 = 0,\ p_1 \to 0\ (y_1 \to \infty),\\ \partial p_1/\partial y_1 = \gamma_5 \partial^2 D/\partial x_3^2 \quad (y_1 = 0),\\ \gamma_5 = \lambda/2\pi A^2 K l^6,\ 0 \leqslant \gamma_5 \leqslant 1,\ p_1(x_3, y_1, z_3) = p_1(x_3, y_1, z_3 + 2\pi),\\ \partial u_3/\partial x_3 + \partial v_3/\partial y_3 + \partial w_3/\partial z_3 = 0,\ \partial p_3/\partial y_3 = 0,\\ u_3\partial u_3/\partial x_3 + v_3\partial u_3/\partial y_3 + w_3\partial u_3/\partial z_3 = \partial^2 u_3/\partial y_3^2,\\ u_3\partial w_3/\partial x_3 + v_3\partial w_3/\partial y_3 + w_3\partial w_3/\partial z_3 + \partial p_3/\partial z_3 = \partial^2 w_3/\partial y_3^2,\end{gathered}$$

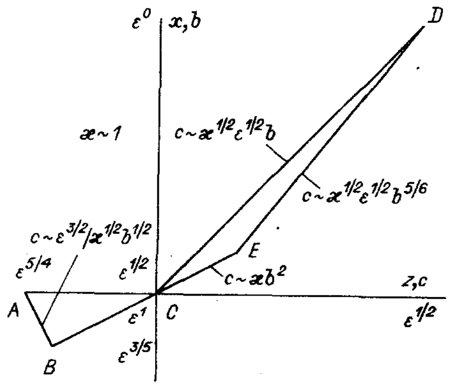

Fig. 6

$$
\begin{gathered}
u_3 = v_3 = w_3 = 0 \qquad (y_3 = 0), \\
u_3 \to y_3,\ v_3,\ w_3,\ p_3,\ D \to 0 \qquad (x_3 \to -\infty), \\
u_3 \to y_3 + \gamma_4 D,\ w_3 \to 0 \qquad (y_3 \to \infty), \\
p_3 = D + p_1(x_3, 0, z_3),\ \gamma_4 = \lambda^2/4\pi^2 K l^5,\ 0 \leqslant \gamma_4 \leqslant 1, \\
u_3,\ v_3,\ w_3(x_3,\ y_3,\ z_3) = u_3,\ v_3, \\
w_3(x_3,\ y_3,\ z_3 + 2\pi), \\
p_3,\ D(x_3,\ z_3) = p_3,\ D(x_3, \\
z_3 + 2\pi).
\end{gathered}
\tag{6.7}
$$

Here the indices 1 and 3 in the variables denote the numbers of the region; $D(x_3, z_3)$ is the thickness of the expulsion of region 3; the parameters γ_4 and γ_5 characterize the degree of interactions of the regions 2-3 and 1-2, respectively; and ℓ is the thickness of the region 3. Linearizing the problem (6.7) for region 3 relative to the initial conditions and using (2.9) and the representation $D(x_3, z_3) = D_1\exp(\beta x_3)\cos z_3$ we can reduce the problem entirely to a system or ordinary differential equations and boundary conditions:

$$
\begin{gathered}
\beta U_1 + dV_1/dy_3 + W_1 = 0, \quad \beta y_3 U_1 + V_1 = d^2U_1/dy_3^2, \\
\beta y_3 W_1 - D_1(1 - \gamma_5\beta^2) = d^2W_1/dy_3^2, \\
U_1(0) = V_1(0) = W_1(0) = W_1(\infty) = 0,\ U_1(\infty) = \gamma_4 D_1,
\end{gathered}
$$

the solution of which is expressed in terms of the Airy function $\mathrm{Ai}(\eta)$ and has the dispersion relation

$$
\gamma_5\beta^2 - 3\gamma_4\beta^{5/3} d\,\mathrm{Ai}\,(0)/d\eta = 1. \tag{6.8}
$$

It follows from Eq. (6.8) that for $\gamma_4 = 0$, $\gamma_5 = 1$, $\ell = (\lambda/2\pi A^2K)^{1/6}$ (when the regions 2 and 3 do not interact) we have $\beta = 1$. The corresponding limiting solution can be obtained from Eq. (4.4) with $\gamma_1 = \gamma_2 = 1$, $\gamma_3 = 2\pi\delta_1/\lambda$, $0 < \gamma_3 \leq 1$. In the linear approximation, using (2.9) and the profile (4.5) with $\gamma_1 = 1$, the problem can be solved analytically [22] and the following expression is obtained for β:

$$
\gamma_3/\beta_n^2 = (n^2 - 1)/2 + n\gamma_3, \quad n = 1, 2, 3, \ldots \tag{6.9}
$$

It is easy to see that as γ_1, $\gamma_3 \to 1$ the formulas (4.6) and (6.9) become identical and only $\beta_1 \to 1$ as $\gamma_3 \to 0$; for all higher order modes $\beta_n \sim \gamma_3^{1/2} \to 0$, $n > 1$.

In the other case, when the region 1 is not perturbed, $\gamma_5 = 0$, $\gamma_4 = 1$, $\ell = (\lambda^2/4\pi^2K)^{1/5}$ and it follows from Eq. (6.8) that $\beta = (-3d\ \mathrm{Ai}(0)/d\eta)^{-3/5} \approx 1.165$. The limiting solution can be obtained numerically from Eq. (5.8) for the profile (4.5) with $\gamma_1 = 1$ [21]. Figure 5 shows β_1 for the first mode versus Re_2 (curve 1). One can see that for $\mathrm{Re}_2 \approx 0.486$ this mode becomes neutral. As Re_2 increases, β_1 approaches its asymptotic value.

Figure 6 shows a diagram of the development of Görtler vortices in the boundary layer of a liquid near a concave surface. Here the width c of the vortex region is plotted along the abscissa axis and the extent of the region b is plotted along the ordinate axis; $\varkappa \sim 1$. The point B corresponds to the regime described by the solution of the boundary-value problem (2.8). As the values of c and λ increase the vortices "float up". This transition corresponds to the line AB and the boundary-value problem (3.4), and here the short-wavelength vortices can become neutral. As the transverse dimensions of the vortices increase, the extent b changes in accordance with the line BC. At the point C the vortices fill the entire thickness of the boundary layer and their development is described by the solution of the boundary-value problem (4.4).

As c and λ increase further the first mode separates from all subsequent modes. The higher order modes are localized inside the boundary layer (the line CD, boundary-value problem (5.6)), and for them $c > \boldsymbol{a}$. The first long-wavelength mode first develops along the line CE with $\boldsymbol{a} \sim c > \delta$. Beyond the point E (boundary-value problem (6.7)) the vortex region starts to flatten and at the point D all modes once again merge; here the long-wavelength vortices can become neutral.

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